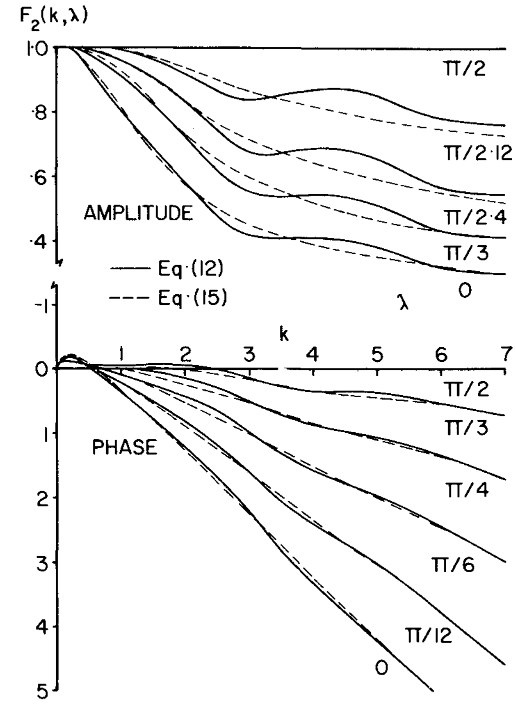

Fig. 2 Obliqueness factor: $F_2(k, \lambda)$.

mations should, of course, reduce to the correct asymptotic forms in limiting cases.

An excellent approximation to the Sears function is[8]

$$S(k) \approx [(k+0.1811)/(0.1811+1.569k+2\pi k^2)]^{1/2} \times \exp\{ik[1-(\tfrac{1}{2}\pi^2)/(1+2\pi k)]\} \quad (13)$$

In very few applications (i.e., calculation of $\dot{\alpha}$ derivatives[9]) this expression's failure to exactly account for the Sears functions logarithmic behavior as $k \to 0$ may lead to inconsistencies.

The following asymptotically correct expressions (indicated by broken lines on Figs. 1 and 2) may be used to replace Eqs. (10) and (11)

$$F_1(r,\theta) \approx \left[\frac{4\pi+2\cos^2\theta r^3+\pi\sin\theta r^4}{4\pi+\pi\cos^2\theta r^2+\pi r^4}\right]^{1/2} \times \exp\left\{-ir\cos\theta\left[\frac{\frac{1}{2}(\pi/2-\theta)+r\cos\theta}{(\pi/2-\theta)+r\cos\theta}\right]\right\} \quad (14)$$

$$F_2(k,\lambda) \approx \left[\frac{2\pi+r^3}{1+\pi r^2+\frac{1}{2}\pi r^4}\,\frac{2\pi+\pi\sin^2\lambda r^2+\frac{1}{2}\pi r^4}{2\pi+\sin^2\lambda r^3+\frac{1}{2}\pi\cos\lambda r^4}\right]^{1/2} \times \exp[\![ik\{1-\sin\lambda-\pi^2/(2+4\pi k)+\tfrac{1}{2}\pi\lambda(2+\cos\lambda)/[1+\pi k(2+\cos\lambda)]\}]\!] \quad (15)$$

Equation (14) with $\theta = 0$ can be used to form a convenient algebraic approximation to Osborne's result.[2] Equation (15) incorporates a better approximation to Eq. (12) than the one suggested (mainly because of its simplicity) in Ref. 7 and used in Ref. 3. Combination of Eq. (15) with the exact expression for the Sears function gives the asymptotically correct linearized incompressible result for all extreme values of k_1 and k_2.

Conclusions

These results, with Eqs. (14) and (15), provide a formula that interpolates between the known analytical expressions for the oblique compressible Sears function. With $M = 0$ the exact linearized result is recovered for all k_1 if $k_2 \to 0$ or $k_2 \to \infty$. In subsonic flow nonzero values to k_1 and/or k_2 reduce the lift coefficient; mitigating the increase in surface velocities accompanying increasing Mach number in steady, two-dimensional flow and delaying the onset of transonic effects. Computed values indicate that the increase in lift curve slope with increasing M is reversed (that is T_A becomes less than a_o) if $M > 2k_1^{1/2}$. Correspondence with accurate numerical solutions of the usual linearized formulation is not necessarily the best basis for determining limits of applicability. Validity of the usual formulation at high frequency (large k_1) may be questioned with regard to both linearization of the potential equation[10] and application of the Kutta condition.[11] Indirect experimental verification could ensue from spectral analysis of airfoil response to grid turbulence or other unsteady inputs.

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Extension of the Momentum Integral Method to Three-Dimensional Viscous-Inviscid Interactions

M. P. GEORGEFF*

La Trobe University, Bundoora, Victoria, Australia

Introduction

A NUMBER of methods have been advanced for investigating two-dimensional laminar viscous-inviscid interactions. One of the most successful is the method of Lees and Reeves[1] as developed by Klineberg[2] and Georgeff.[3]

The development of methods capable of predicting the main features of three-dimensional viscous-inviscid interactions, however, has been limited. Leblanc, Horton, and Ginoux[4] developed a method based on that of Lees and Reeves for investigating axisymmetric flow with spin. However, their method was limited

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* Research Fellow, School of Education.

to adiabatic flows and required the determination of a large number of additional profile functions.

In this Note the Lees-Reeves method as modified by Georgeff is extended to include three-dimensional adiabatic and non-adiabatic flows with zero transverse flow gradient (i.e., flows over infinite swept wings and rotating axisymmetric bodies). Only two new profile functions are required and all can be determined from the Cohen-Reshotko similarity solutions for two-dimensional flow.[5]

II. Three-Dimensional Plane Flow

In Stewartson coordinates,[6] the governing equations for a steady three-dimensional compressible laminar boundary layer with zero transverse flow gradient are

Continuity

$$U_X + V_Y = 0 \tag{1}$$

Longitudinal Momentum

$$UU_X + VU_Y = U_e(U_e)_X\{1 + S_w s - \omega g^2\} + v_\infty U_{YY} \tag{2}$$

Transverse Momentum

$$Ug_X + Vg_Y = v_\infty g_{YY} \tag{3}$$

Energy

$$Us_X + Vs_Y = v_\infty s_{YY} \tag{4}$$

where X and Y are the transformed coordinates; U and V are the transformed velocity components;

$$g = [1-(w/w_e)]; \quad s = S/S_w$$

$$S = [(h_0)_{tr}/(h_{0_e})_{tr}] - 1; \quad \omega = w_e^2/2(h_{0_e})_{tr} = \text{const}$$

and where the subscript "tr" denotes values determined with respect to a coordinate frame moving in the z direction with velocity w_e, i.e.

$$(h_0)_{tr} = h + [u^2 + (w-w_e)^2]/2$$

The boundary conditions for "g" and "s" are

$$g(X,0) = s(X,0) = 1$$
$$\lim_{Y\to\infty}(g) = \lim_{Y\to\infty}(s) = 0 \tag{5}$$

As the same Eqs. (3) and (4) and boundary conditions (5) apply to both "g" and "s," a particular solution for "g" is

$$g = s \tag{6}$$

Then Eq. (2) reduces to

$$UU_X + VU_Y = U_e(U_e)_X\{1 + S_w(s - k_1 s^2)\} + v_\infty U_{YY} \tag{7}$$

where $k_1 = \omega/S_w$.

By integrating across the boundary layer the following integral forms can be obtained

Continuity

$$\bar{F}\frac{d\ln\delta_i^*}{dx} + \frac{dH}{dx} + \frac{1+m_e}{m_e}S_w\frac{d\bar{E}}{dx} + \bar{J}\frac{d\ln M_e}{dx} = \frac{B(1+m_e)\tan(\theta_e - \alpha_w)}{m_e(1+m_\infty)\delta_i^*} \tag{8}$$

Longitudinal Momentum

$$H\frac{d\ln\delta_i^*}{dx} + \frac{dH}{dx} + (2H + 1 + S_w\bar{E})\frac{d\ln M_e}{dx} = BC\frac{M_\infty}{M_e}\frac{P}{\delta_i^* Re_{\delta_i^*}} \tag{9}$$

Longitudinal Moment of Momentum

$$J\frac{d\ln\delta_i^*}{dx} + \frac{dJ}{dx} + (3J + 2S_w\bar{T}^*)\frac{d\ln M_e}{dx} = BC\frac{M_\infty}{M_e}\frac{R}{\delta_i^* Re_{\delta_i^*}} \tag{10}$$

Energy

$$T^*\frac{d\ln\delta_i^*}{dx} + \frac{dT^*}{dx} + T^*\frac{d\ln M_e}{dx} = BC\frac{M_\infty}{M_e}\frac{Q}{\delta_i^* Re_{\delta_i^*}} \tag{11}$$

where

$$B = (a_e p_e)/(a_\infty p_\infty); \quad m = 1/2(\gamma-1)M^2$$

θ_e is the streamline inclination at the edge of the boundary layer α_w is the local surface slope, and the transformed profile quantities are

$$\delta_i = \int_0^{\delta_i} dY \qquad \delta_i^* = \int_0^{\delta_i}\left(1 - \frac{U}{U_e}\right)dY$$

$$H = \frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right)dY \qquad J = \frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{U}{U_e}\left(1 - \frac{U^2}{U_e^2}\right)dY$$

$$P = \delta_i^*\frac{\partial}{\partial Y}\left(\frac{U}{U_e}\right)_{Y=0} \qquad R = \delta_i^*\int_0^{\delta_i}\frac{\partial}{\partial Y}\left(\frac{U}{U_e}\right)^2 dY$$

$$Z = \frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{U}{U_e}dY \qquad Q = -\delta_i^*\frac{\partial}{\partial Y}\left(\frac{S}{S_w}\right)_{Y=0}$$

$$E = \frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{S}{S_w}dY \qquad G = \frac{1}{\delta_i^*}\int_0^{\delta_i}\left(\frac{S}{S_w}\right)^2 dY$$

$$T^* = \frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{U}{U_e}\frac{S}{S_w}dY \qquad W^* = \frac{1}{\delta_i^*}\int_0^{\delta}\frac{U}{U_e}\left(\frac{S}{S_w}\right)^2 dY$$

$$\bar{E} = E - k_1 G \qquad \bar{T}^* = T^* - k_1 W^*$$

$$\bar{F} = H + \frac{1+m_e}{m_e}(1 + S_w\bar{E})$$

$$\bar{J} = \frac{1}{1+m_e}\left[m_e\bar{F} + 2H - \frac{1+2m_e}{m_e}Z\right] - \left[\bar{F} + \frac{Z}{m_e}\right]\frac{M_e}{p_e}\frac{dp_e}{dM_e}$$

Following the procedure used for two-dimensional flows,[1] the similarity solutions are employed to provide the required universal relations between profile quantities. Equations (1, 3, 4, and 7) can be reduced to similarity form by letting

$$U_e = C_1 X^n$$

and taking as a similarity variable

$$\eta = Y\left|\frac{n+1}{2}\frac{U_e}{v_\infty X}\right|^{1/2}$$

The solution to these equations yields a set of longitudinal velocity and total enthalpy profiles for each value of S_w and k_1. These profiles are then decoupled and two independent parameters, one representing the longitudinal velocity profile and the other the total enthalpy profile, are introduced. No parameter is required for the transverse velocity profile as it is equivalent to the corresponding total enthalpy profile. The longitudinal velocity profile parameter is taken to be the profile quantity "H" and the total enthalpy profile parameter, normalized with respect to S_w, is taken to be

$$b = -(\partial/\partial\eta)(S/S_w)_{\eta=0}$$

Arbitrarily taking the edge of the boundary layer (η_e) at $(u/u_e) = 0.99$, all the profile quantities appearing in the governing differential equations can be expressed as functions of these two parameters for any particular values of S_w and k_1. The determination of the functions that depend on both the velocity and enthalpy profiles can be simplified by using the integrals

$$\alpha(H) = \left[\int_0^{\eta_{0.99}}\left(1 - \frac{U}{U_e}\right)d\eta\right]^{-1}$$

$$\sigma(b) = \int_0^{\eta_{0.99}}\frac{S}{S_w}d\eta \qquad \kappa(b) = \int_0^{\eta_{0.99}}\left(\frac{S}{S_w}\right)^2 d\eta$$

$$T(H,b) = \int_0^{\eta_{0.99}}\frac{U}{U_e}\frac{S}{S_w}d\eta \qquad W(H,b) = \int_0^{\eta_{0.99}}\frac{U}{U_e}\left(\frac{S}{S_w}\right)^2 d\eta$$

such that $Q = b/\alpha(H)$

$$E = \alpha(H)\sigma(b) \qquad G = \alpha(H)\kappa(b)$$

$$T^* = \alpha(H)T(H,b) \qquad W^* = \alpha(H)W(H,b)$$

Now for $k_1 = 0$, it has been shown[3] that the profile functions $J(H)$, $T(H,b)$, etc., are virtually independent of S_w. Similarly, Leblanc, Horton, and Ginoux[4] have indicated that for $k_1 = \frac{1}{2}$, these functions are again independent of S_w (or ω). It is therefore reasonable to assume that the profile functions are independent of S_w and ω for all values of k_1, provided S_w and ω are of order unity or less. The reason for this independence is that the sets of velocity and total enthalpy profiles obtained at different values of S_w and k_1 form, to a good approximation, a one parameter family of solutions, i.e., a change in S_w or k_1 has a similar effect as a related change in the pressure gradient parameter β which, although affecting all the profiles, does not affect the relations between profile quantities.

Using the normalized form of the profile quantities, the solutions to the similarity equations at any particular value of S_w and ω therefore provide a set of universal relations that are valid for all S_w and ω. The profile functions applicable for three-dimensional flow may thus be obtained from the Cohen-Reshotko[5] similarity solution for two-dimensional flow (i.e., where $\omega = 0$). This is of considerable gain, as the solutions are readily available and the onerous task of recalculating the profile functions at each value of S_w and ω is avoided.

In order to make the method amenable to numerical techniques, the profile functions thus obtained are finally expressed in the form of polynomial functions of the parameters H and b, using a least squares curve fit.[2,3]

The final form of the governing differential equations becomes

$$\bar{F}\frac{d\ln\delta_i^*}{dx} + \frac{\partial\bar{F}}{\partial H}\frac{dH}{dx} + \frac{\partial\bar{F}}{\partial b}\frac{db}{dx} + \bar{J}\frac{d\ln M_e}{dx} = \frac{B(1+m_e)}{m_e(1+m_\infty)}\frac{\tan(\theta_e-\alpha_w)}{\delta_i^*} \quad (12)$$

$$H\frac{d\ln\delta_i^*}{dx} + \frac{dH}{dx} + (2H+1+S_w\bar{E})\frac{d\ln M_e}{dx} = BC\frac{M_\infty}{M_e}\frac{P}{\delta_i^* Re_{\delta_i^*}} \quad (13)$$

$$J\frac{d\ln\delta_i^*}{dx} + \frac{dJ}{dH}\frac{dH}{dx} + (3J+2S_w\bar{T}^*)\frac{d\ln M_e}{dx} = BC\frac{M_\infty}{M_e}\frac{R}{\delta_i^* Re_{\delta_i^*}} \quad (14)$$

$$T^*\frac{d\ln\delta_i^*}{dx} + \frac{\partial T^*}{\partial H}\frac{dH}{dx} + \frac{\partial T^*}{\partial b}\frac{db}{dx} + T^*\frac{d\ln M_e}{dx} = BC\frac{M_\infty}{M_e}\frac{Q}{\delta_i^* Re_{\delta_i^*}} \quad (15)$$

These equations are similar in form to the equations for two-dimensional flow[3] and the same solution procedure may thus be employed. The model for the outer inviscid flow, which specifies the relationship between the streamline inclination at the edge of the boundary layer θ_e, local static pressure p_e and Mach number M_e can be provided by any of the methods common to inviscid flow theory.

III. Axisymmetric Flow with Spin

The results of the preceding section can be readily generalized to include axisymmetric flow. It is assumed that $\delta \ll r_w$ and the following quantities are redefined

$$g = w/w_w = w/(\Omega r_w), \qquad \omega = w_w^2/2h_{0_e}$$

where r_w is the local radius of the body and Ω is the angular velocity of the body.

For k_1 constant, i.e.,

$$\frac{1}{S_w}\frac{dS_w}{dx} = \frac{1}{\omega}\frac{d\omega}{dx} = \frac{2}{r_w}\frac{dr_w}{dx}$$

then

$$g = s \quad (16)$$

and the governing equations reduce to those for three-dimensional plane flow [Eqs. (12–15)] with additional right-hand side terms, respectively,

$$-\left[\frac{2S_w(1+m_e)}{m_e}\bar{E} - \frac{Z}{m_e}\right]\frac{d\ln r_w}{dx} \quad (12')$$

$$-\left[H + \frac{S_w(1+m_e)}{m_e}k_1 G\right]\frac{d\ln r_w}{dx} \quad (13')$$

$$-\left[J + \frac{2S_w(1+m_e)}{m_e}k_1 W^*\right]\frac{d\ln r_w}{dx} \quad (14')$$

$$-3T^*\frac{d\ln r_w}{dx} \quad (15')$$

The profile functions are unchanged as they are obtained from the similarity solutions with $dr_w/dx = 0$.

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Linear Spatial Stability of the Plane Poiseuille Flow

R. K. Leong*
National Research Council of Canada, Ottawa, Canada

THE linear stability of the plane Poiseuille flow has received much attention in both theory and experiment. Many of the investigators[1,2] worked out analytical solutions by assuming that the periodic disturbances were subject to amplification in time. This, however, is not a suitable model for the disturbances investigated experimentally, which are quasi-steady and vary in amplitude with distance downstream. A better model is obtained by considering a disturbance travelling in the direction of flow and having a nondimensional stream function of the form

$$\psi = \phi(y)\,e^{i(\alpha x-\beta t)} + \tilde{\phi}(y)\,e^{-i(\tilde{\alpha}x-\beta t)} \quad (1)$$

where the frequency β is real and the symbol ˜ denotes a complex conjugate. The amplitude of the perturbation $\phi(y)$ is assumed to be small. The substitution of ψ in the nondimensional linearized vorticity equation for the perturbation leads to the Orr-Sommerfeld equation

$$(\bar{u}-\beta/\alpha)(\phi''-\alpha^2\phi)-\bar{u}''\phi+(i/\alpha R)(\phi''''-2\alpha^2\phi+\alpha^4\phi)=0 \quad (2)$$

where an accent (′) indicates differentiation with respect to y, R is the Reynolds number, and $\bar{u} = 1-y^2$ is the nondimensional mean flow velocity. This equation, together with its boundary conditions

$$\phi(y) = 0, \quad \phi'(y) = 0 \quad \text{at} \quad y = \pm 1 \quad (3)$$

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* Computation Officer, High Speed Aerodynamics Laboratory.